

STRUCTURE OF SHOCKS IN BURGERS TURBULENCE WITH LÉVY NOISE INITIAL DATA

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ABSTRACT. We study the structure of the shocks for the inviscid Burgers equation in dimension 1 when the initial velocity is given by Lévy noise, or equivalently when the initial potential is a two-sided Lévy process ψ_0 . When ψ_0 is abrupt in the sense of Vigon or has bounded variation with $\limsup_{|h|\downarrow 0} h^{-2}\psi_0(h) = \infty$, we prove that the set of points with zero velocity is regenerative, and that in the latter case this set is equal to the set of Lagrangian regular points, which is non-empty. When ψ_0 is abrupt we show that the shock structure is discrete. When ψ_0 is eroded we show that there are no rarefaction intervals.

1. INTRODUCTION

Burgers introduced the equation

$$\partial_t u + \partial_x(u^2/2) = \varepsilon \partial_{xx}^2 u$$

as a simple model of hydrodynamic turbulence for compressible fluids, where the parameter $\varepsilon > 0$ describes the viscosity of the fluid and the solution represents the velocity of a fluid particle located at x at time t [8]. It can be seen as a simplification of the Navier-Stokes equation arrived at by neglecting pressure and force terms, but also arises in other physical problems, such as the formation of the superstructure of the universe [29].

It is known that under certain conditions, as $\varepsilon \rightarrow 0$ the solution converges to the unique entropy condition satisfying weak solution of the *inviscid Burgers equation*

$$(1.1) \quad \partial_t u + \partial_x(u^2/2) = 0.$$

A physical interpretation of the weak entropy condition satisfying solution to (1.1) is that at time zero, infinitesimal particles are uniformly spread on the line, with initial velocity $u(\cdot, 0)$, and these particles evolve according to the dynamics of completely inelastic shocks. That is, the velocity of a particle changes only when the cluster of particles it is in collides with another cluster, in which case the clusters stick together and form a heavier cluster, with conservation mass and momentum determining the mass and velocity of the new cluster.

There is an abundant literature on the solution to 1.1 when the initial velocity $u(\cdot, 0)$ is a random process. See for example [3, 2, 13, 12, 7, 6, 8, 9, 15, 21, 22, 23, 25, 29, 16, 28]. We will investigate the solution when $u(\cdot, 0)$ is a Lévy noise, i.e. when the *potential* process $\psi_0 = (\psi_0(x))_{x \in \mathbb{R}}$, defined by $\psi_0(x) - \psi_0(y) = \int_x^y u(z, 0) dz$, has stationary independent increments. In particular, we investigate

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qualitative features of the shock structure of the solution, and thus extend the work of Bertoin [7] (ψ_0 a stable Lévy process with stability index $\alpha \in (1/2, 2]$), Giraud [13] (extensive results for the case $\alpha \in (1/2, 1)$) and Lachièze-Rey [16] (ψ_0 a bounded variation Lévy process).

In order to explain our results, we must first discuss the general solution to (1.1) and some related concepts. We follow [13, Section 2.1] closely. Suppose that ψ_0 has discontinuities only of the first kind and satisfies $\psi_0(x) = o(x^2)$ as $|x| \rightarrow \infty$. Then as $\varepsilon \rightarrow 0$ the unique solution of Burgers equation with viscosity $\varepsilon > 0$ converges (except on a countable set) to a weak solution of (1.1), referred to as the Hopf-Cole solution (see [14, 10]). The right continuous version of this solution is

$$u(x, t) = t^{-1}(x - a(x, t)),$$

where, taking the supremum over all possible arguments if necessary,

$$a(x, t) := \arg \sup \left\{ \psi_0(y) - \frac{1}{2t}(y - x)^2 : y \in \mathbb{R} \right\}.$$

The function $x \mapsto a(x, t)$ is non-decreasing and right continuous and its right continuous inverse $a \mapsto x(a, t)$ is known as the *Lagrangian function*, and gives the position at time t of the particle initially located at a .

A discontinuity of $x \mapsto u(x, t)$ is called a shock and occurs when $x \mapsto a(x, t)$ jumps, i.e. when $a(x, t) \neq a(x-, t) := \lim_{y \uparrow x} a(y, t)$. From the point of view of the particle description, the location of a shock corresponds to the location of a cluster at time t . This cluster results from the aggregation of the particles initially located in $[a(x-, t), a(x, t)]$; its velocity is (according to the conservation of masses and momenta)

$$v(x, t) = -\frac{\psi_0(a(x, t)) - \psi_0(a(x-, t))}{a(x, t) - a(x-, t)} = \frac{1}{2} [u(a(x, t)) + u(a(x-, t))].$$

The interval $[a(x-, t), a(x, t)]$ is called a *shock interval* and x a *Eulerian shock point*. We define the *shock structure* of the solution at time t to be the closed range of $a(\cdot, t)$. Of particular interests are points which are not isolated on the left or the right in that closed range, since they represent the initial locations of particles that have not been involved in any collisions by time t . We call any such point a *Lagrangian regular point*. Finally, we call (x, y) a *rarefaction interval* if $a(\cdot, t)$ stays constant on $[x, y)$. A rarefaction interval represents an interval where there are no fluid particles at time t .

Our results concern qualitative features of the shock structure, the regenerativity of the process $(u(x, t))_{x \in \mathbb{R}}$ at points where $u(x, t) = 0$, and the relationship between such points and the Lagrangian regular points. For our arguments, there is no loss of generality to assume $t = 1$ – the properties we show will be true for any $t > 0$. Thus we restrict our attention to the case $t = 1$ and set $a(x) = a(x, 1)$, $u(x) = u(x, 1)$ for all $x \in \mathbb{R}$. The shock structure is then

$$\mathcal{A} := \text{cl}\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\},$$

i.e. the closure of the range of $a(\cdot)$, and Lagrangian regular points are the subset of points of \mathcal{A} that are neither left nor right isolated. We also define $\mathcal{A}_0 \subset \mathcal{A}$ by

$$\mathcal{A}_0 := \text{cl}\{x \in \mathbb{R} : a(x) = x\} = \text{cl}\{x \in \mathbb{R} : u(x) = 0\},$$

Note that both \mathcal{A} and \mathcal{A}_0 are stationary sets when ψ_0 is a Lévy process, and since adding a drift term has no affect on the distributions of these random sets, *we will assume throughout that if ψ_0 has bounded variation then it has zero drift coefficient.*

To ensure that \mathcal{A} is non-empty we will always assume that $\lim_{|x| \rightarrow \infty} x^{-2}\psi_0(x) = 0$, and in the bounded variation case we mostly assume that $\limsup_{|h| \downarrow 0} h^{-2}\psi_0(h) = \infty$ to ensure that \mathcal{A} has a nice structure. Most of our results in the bounded variation case also require a further assumption relating to overshoots at hitting times – see Assumption **B** in Section 3.3.

In all cases we show that the Lebesgue measure of $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ is zero (see Lemma 4.1) and in the bounded variation case we show that this set is closed (see Theorem 4.14). For ψ_0 in an interesting class of unbounded variation Lévy processes called *abrupt* Lévy processes (see Section 3.4 for a definition), we also show that this set is closed and that moreover \mathcal{A} is a discrete set (see Theorem 4.6 and Corollary 4.7), extending the result of [7] that this is true when ψ_0 is a stable process with $\alpha \in (1, 2]$. A result from [7] relating to Cauchy processes is also extended to a more general class of unbounded variation processes, the *eroded* Lévy processes (again, see Section 3.4 for a definition). For these eroded processes, there are no rarefaction intervals.

We show that if ψ_0 is of unbounded variation and abrupt, or of bounded variation and satisfying Assumption **B**, then the process $u = (u(x))_{x \in \mathbb{R}}$ is regenerative at points y such that $u(y) = 0$, that between any two consecutive such points it must first be positive and then negative, and that the only accumulation points of jump times of u are at such points (see Theorem 4.3, Theorem 4.6, Proposition 4.12 and Theorem 4.15). For ψ_0 a stable processes with $\alpha \in (1/2, 1)$, this is the main result of [13], hence our work generalizes that result to a wider class of bounded variation processes (it is also shown in [13] that for those stable processes \mathcal{A} is a discrete set – we could not generalize this result to our wider class of bounded variation processes). Key to proving this result is the theory of *randomized coterminial times* due to Millar (see Section 3.5), which allows us to decompose the process at $T := \inf\{x \geq 0 : x \in \mathcal{A}_0\}$, i.e. at the first non-negative element of \mathcal{A}_0 . The results of Lachièze-Rey [16] also form an indispensable part of our arguments in the bounded variation case.

Another important result of [13] is that when ψ_0 is a stable processes with $\alpha \in (1/2, 1)$, \mathcal{A}_0 is exactly equal to the set of points of \mathcal{A} at which ψ_0 is continuous, which is in turn equal to the set of Lagrangian regular points. We extend this result to our more general class of bounded variation processes (see Proposition 4.12 and Theorem 4.15) again using the results of Lachièze-Rey [16].

The rest of the paper is organized as follows. In Section 2 we discuss geometric interpretations of $a(x)$ that make the proofs easier to read, and introduce the important connection between \mathcal{A} and the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x \in \mathbb{R}}$. In Section 4 we present and prove all of our results, with the exception of the proof of the regenerativity property of \mathcal{A}_0 mentioned above, which we prove in Section 5.

We conclude the introduction by noting that \mathcal{A}_0 is the set of fixed points of the proximal mapping for the Moreau envelope of ψ_0 [17, 20] and thus may be of interest in convex analysis.

2. GEOMETRIC INTERPRETATIONS AND RELATION TO CONCAVE MAJORANTS

Recall from Section 1 that

$$a(x) = \arg \sup \left\{ \psi_0(y) - \frac{1}{2}(y-x)^2 : y \in \mathbb{R} \right\},$$

i.e. $a(x)$ is the (largest) location of the supremum of $y \mapsto \psi_0(y) - \frac{1}{2}(y-x)^2$. One has the following geometric interpretation: consider a realization of the initial potential ψ_0 and a parabola $y \mapsto \frac{1}{2}(z-x)^2 + C$, where C is chosen such that the parabola is strictly above the path of ψ_0 . Let C decrease until this parabola touches the graph of ψ_0 . Then $a(x)$ is the largest abscissa of the contact points.

Now consider what happens to $a(x)$ as x increases. Suppose for example that $x < a(x)$, then the center of the parabola will move forward, and C will increase so that the largest abscissa of the contact points between the parabola and ψ_0 remains at $a(x)$. This will keep going until for some $z > x$, the location of the largest supremum of $y \mapsto \psi_0(y) - \frac{1}{2}(y-z)^2$ is no longer at $a(x)$, that is, the parabola centered at z passing through the point $(a(x), \psi_0(a(x)))$ will touch ψ_0 again at $(a(z), \psi_0(a(z)))$, where $a(z) > a(x)$. This creates a jump in a , and hence in u , at the location z . The story is similar when $x > a(x)$, except that now C will decrease in order to keep the parabola touching ψ_0 as the center of the parabola moves forward.

Another important geometric property of the Hopf-Cole solution relates to concave majorants. For any $f : \mathbb{R} \rightarrow \mathbb{R}$, the concave majorant of f is the minimal concave function $\bar{C}_f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ such that $\bar{C}_f(x) \geq f(x) \vee f(x-)$ for every $x \in \mathbb{R}$.

Let $\bar{C} : \mathbb{R} \rightarrow \mathbb{R}$ denote the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x \in \mathbb{R}}$, and denote its right continuous derivative by $\bar{c} = \bar{C}'$. Since $\bar{c}(\cdot)$ is non-increasing, we can consider the Stieltjes measure $-d\bar{c}$. The connection with \mathcal{A} is the following.

Lemma 2.1. *For any ψ_0 ,*

- (i) $\text{Supp}(d\bar{c}) \subseteq \mathcal{CL}\{y : \exists x \text{ s.t. } a(x) = y\} = \mathcal{A}$;
- (ii) $\{y : \exists x \text{ s.t. } a(x) = y\} \subseteq \text{Supp}(d\bar{c})$.

Hence if $\{y : \exists x \text{ s.t. } a(x) = y\}$ is closed then $\text{Supp}(d\bar{c}) = \mathcal{A}$.

Proof. (i) Suppose first that $y \in \text{Supp}(d\bar{c})$ is isolated on both sides in $\text{Supp}(d\bar{c})$ or is in the interior of $\text{Supp}(d\bar{c})$. Then there exists $x \in \mathbb{R}$ such that

$$(\psi_0(y+z) \vee \psi_0((y+z)-) - \frac{1}{2}(y+z)^2) - (\psi_0(y) \vee \psi_0(y-) - \frac{1}{2}y^2) < -xz$$

for all $z \neq 0$. But then

$$\psi_0(y+z) \vee \psi_0((y+z)-) - \frac{1}{2}(y+z-x)^2 \leq \psi_0(y) \vee \psi_0(y-) - \frac{1}{2}(y-x)^2$$

with equality only if $z = 0$. Hence

$$a(x) = y + \arg \sup \left\{ \psi_0(y+z) - \frac{1}{2}((y+z)-x)^2 : z \in \mathbb{R} \right\} = y + 0 = y,$$

and thus $y \in \mathcal{A}$.

Now suppose y is not isolated in $\text{Supp}(d\bar{c})$. Then there exists a sequence of points $\{y_n\}_{n \geq 0}$ such that $y_n \rightarrow y$ with each y_n either isolated on both sides in $\text{Supp}(d\bar{c})$ or in the interior of $\text{Supp}(d\bar{c})$. Let $\{x_n\}_{n \geq 0}$ be such that $a(x_n) = y_n$ for each $n \geq 0$. Then $a(x_n) \rightarrow y$ and hence $y \in \mathcal{A}$ since \mathcal{A} is closed.

(ii) Suppose there exists x such that $a(x) = y$. From the definition of $a(x)$ it follows that

$$\psi_0(y-z) \vee \psi_0((y-z)-) - \frac{1}{2}((y-z)-x)^2 \leq \psi_0(y) \vee \psi_0(y-) - \frac{1}{2}(y-x)^2$$

for all $z \geq 0$ and

$$\psi_0(y+z) \vee \psi_0((y+z)-) - \frac{1}{2}((y+z)-x)^2 < \psi_0(y) \vee \psi_0(y-) - \frac{1}{2}(y-x)^2$$

for all $z > 0$. Thus

$$(2.1) \quad (\psi_0(y-z) \vee \psi_0((y-z)-) - \frac{1}{2}(y-z)^2) - (\psi_0(y) \vee \psi_0(y-) - \frac{1}{2}y^2) \leq zx$$

for all $z \geq 0$ and

$$(2.2) \quad (\psi_0(y+z) \vee \psi_0((y+z)-) - \frac{1}{2}(y+z)^2) - (\psi_0(y) \vee \psi_0(y-) - \frac{1}{2}y^2) < -zx$$

for all $z > 0$.

(2.1) implies that $\bar{c}(y-) \geq -z$ and (2.2) implies that $\bar{c}(y) < -z$. Hence $y \in \text{Supp}(d\bar{c})$. \square

3. DEFINITIONS AND BACKGROUND MATERIAL

3.1. Lévy processes. Let $\psi_0 = (\psi_0(x))_{x \in \mathbb{R}}$ be a real-valued Lévy process. That is, ψ_0 has *càdlàg* sample paths, $\psi_0(0) = 0$, and $\psi_0(y) - \psi_0(x)$ is independent of $(\psi_0(z))_{z \leq x}$ with the same distribution as $\psi_0(y-x)$ for all $x, y \in \mathbb{R}$ with $x < y$.

The Lévy-Khintchine formula says that for $x \geq 0$ the characteristic function of $\psi_0(x)$ is given by $\mathbb{E}[e^{i\theta\psi_0(x)}] = e^{-x\Psi(\theta)}$ for $\theta \in \mathbb{R}$, where

$$\Psi(\theta) = -ic\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta y} + i\theta y 1_{\{|y|<1\}}) \Pi(dy)$$

with $c \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and Π a σ -finite measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge y^2) \Pi(dy) < \infty$. We call σ^2 the *infinitesimal variance* of the Brownian component of ψ_0 and Π the *Lévy measure* of X .

The sample paths of ψ_0 have bounded variation almost surely if and only if $\sigma = 0$ and $\int_{\mathbb{R}} (1 \wedge |y|) \Pi(dy) < \infty$. In this case Ψ can be rewritten as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta y}) \Pi(dy).$$

We call $d \in \mathbb{R}$ the drift coefficient. Recall from the introduction that we will assume $d = 0$ throughout without affecting our results. For full details of these definitions see [5].

3.2. Fluctuation theory. We will often make use of some basic results from fluctuation theory for Lévy processes.

The first is due to Shtatland [24]. If ψ_0 has paths of bounded variation with drift d , then

$$(3.1) \quad \lim_{h \downarrow 0} h^{-1} \psi_0(h) = d \quad \text{a.s.}$$

Since the jump times of ψ_0 form a countable set of stopping times, by the strong Markov property it follows that for all y such that $\psi_0(y) \neq \psi_0(y-)$, i.e. at all jump times y of ψ_0 , we have

$$(3.2) \quad \lim_{h \downarrow 0} h^{-1} (\psi_0(y+h) - \psi_0(y)) = d \quad \text{a.s.}$$

The counterpart of Shtatland's result when ψ_0 has paths of unbounded variation is Rogozin's result

$$(3.3) \quad \liminf_{h \downarrow 0} h^{-1} \psi_0(h) = -\infty \quad \text{and} \quad \limsup_{h \downarrow 0} h^{-1} \psi_0(h) = +\infty \quad \text{a.s.}$$

By the strong Markov property, it again follows that for all y such that $\psi_0(y) \neq \psi_0(y-)$, i.e. at all jump times y of ψ_0 , we have

$$(3.4) \quad \begin{aligned} \liminf_{h \downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) &= -\infty \quad \text{a.s.} \quad \text{and} \\ \limsup_{h \downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) &= +\infty \quad \text{a.s.} \end{aligned}$$

3.3. Hypotheses on ψ_0 . We now define some hypotheses on ψ_0 . We will always assume the first and the second ensures that the shock structure is nice when ψ_0 has paths of bounded variation. Let $\bar{C} : \mathbb{R} \rightarrow \mathbb{R}$ denote the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x \in \mathbb{R}}$, and denote its right continuous derivative by $\bar{c} = \bar{C}'$. Since $\bar{c}(\cdot)$ is non-increasing, we can consider the Stieltjes measure $-d\bar{c}$.

Hypothesis A. Let ψ_0 be such that almost surely $\lim_{|x| \rightarrow \infty} x^{-2}\psi_0(x) = 0$.

Remark 3.1. (i) Hypothesis **A** implies \bar{C} is finite and $\sup_{x \in \mathbb{R}} \{\psi_0(x) - \frac{1}{2}x^2\} < \infty$.

(ii) Hypothesis **A** holds for stable processes with stability index $\alpha \in (1/2, 2]$.

Hypothesis B. If ψ_0 has paths of bounded variation then let ψ_0 be such that

$$(3.5) \quad \limsup_{h \downarrow 0} h^{-2}\psi_0(h) = +\infty \quad \text{and} \quad \liminf_{h \downarrow 0} h^{-2}\psi_0(h) = -\infty \quad \text{a.s.}$$

Remark 3.2. (i) If ψ_0 has paths of unbounded variation then (3.5) always holds by (3.3).

(ii) If ψ_0 has paths of bounded variation then by (3.1) Hypothesis **B** implies that ψ_0 has zero drift coefficient (but we are already assuming this is true throughout). In fact, Bertoin et al. have fully characterized which bounded variation Lévy processes satisfy (3.5) [4, Theorem 3.2] (clearly it is necessary to at least have $\Pi((-\infty, 0)) = \Pi((0, \infty)) = \infty$).

(iii) Hypothesis **B** holds for stable processes with stability index $\alpha \in (1/2, 2]$.

(iv) Again by the strong Markov property, under Hypothesis **B** it follows that for all y such that $\psi_0(y) \neq \psi_0(y-)$,

$$(3.6) \quad \begin{aligned} \limsup_{h \downarrow 0} h^{-2}(\psi_0(y+h) - \psi_0(y)) &= +\infty \quad \text{a.s.} \quad \text{and} \\ \liminf_{h \downarrow 0} h^{-2}(\psi_0(y+h) - \psi_0(y)) &= -\infty \quad \text{a.s.} \end{aligned}$$

The following assumption will be necessary for the advanced results in the bounded variation case. Recall that we have already assumed ψ_0 to have zero drift coefficient.

Assumption B. Suppose ψ_0 has paths of bounded variation.

(I) Let $T = \inf_{x \geq 0} \{\psi_0(x) - bx - \frac{1}{2}x^2 \geq s\}$ for some $b > 0$ and $s > 0$. Then on the set $\{T < \infty\}$ we have $\psi_0(T) - bT - \frac{1}{2}T^2 > s$ almost surely.

(II) Let $T = \inf_{x \geq 0} \{\psi_0(x) + bx - \frac{1}{2}x^2 \leq -s\}$ for some $b > 0$ and $s > 0$. Then on the set $\{T < \infty\}$ we have $\psi_0(T) + bT - \frac{1}{2}T^2 < -s$ almost surely.

Remark 3.3. (i) By quasi-continuity of Lévy processes the conclusion still holds when $\psi_0(x)$ is replaced by $\psi_0(x) \vee \psi_0(x)$ in the definitions of T .

(ii) (II) has an equivalent time reversed version: let $T = \inf_{x \leq 0} \{\psi_0(x) + bx - \frac{1}{2}x^2 \geq s\}$ for some $b > 0$ and $s > 0$. Then $\psi_0(T) + bT - \frac{1}{2}T^2 > s$ a.s.

3.4. Abrupt and eroded Lévy processes. A broad class of unbounded variation Lévy processes of interest has been defined by Vigon [27].

Definition 3.4. A Lévy process ψ_0 is *abrupt* if its paths have unbounded variation and almost surely for all m such that ψ_0 has a local maximum at m ,

$$\liminf_{h \downarrow 0} h^{-1}(\psi_0(m-h) - \psi_0(m)) = +\infty \quad \text{and} \quad \limsup_{h \downarrow 0} h^{-1}(\psi_0(m+h) - \psi_0(m)) = -\infty.$$

Remark 3.5. A Lévy process ψ_0 with paths of unbounded variation is abrupt if and only if

$$(3.7) \quad \int_0^1 x^{-1} \mathbb{P}\{\psi_0(x) \in [ax, bx]\} dx < \infty, \quad \forall a < b,$$

(see [27, Theorem 1.3]). Examples of abrupt Lévy processes include stable processes with stability parameter in the interval $(1, 2]$, processes with non-zero Brownian component, and any processes that creep upwards or downwards. An example of an unbounded variation process that is not abrupt is the symmetric Cauchy process, however this process will be *eroded* in the sense of Definition 3.7.

The following theorem describes the local behavior of an abrupt Lévy process at arbitrary times. This result is an immediate corollary of the more general result [27, Theorem 2.6] once we use the fact that almost surely the paths of a Lévy processes cannot have both points of increase and points of decrease [11].

Theorem 3.6. *Let ψ_0 be a two sided abrupt Lévy process. Then, almost surely for all $x \in \mathbb{R}$, if*

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-)) < \infty \quad \text{and} \quad \limsup_{h \downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x)) < \infty$$

then ψ_0 has a local supremum at x .

At the other end of the scale from abrupt processes are eroded processes, also introduced by Vigon [26].

Definition 3.7. A Lévy process ψ_0 is *eroded* if its paths have unbounded variation and almost surely for all m such that ψ_0 has a local maximum at m ,

$$\liminf_{h \downarrow 0} h^{-1}(\psi_0(m-h) - \psi_0(m)) = 0 \quad \text{and} \quad \limsup_{h \downarrow 0} h^{-1}(\psi_0(m+h) - \psi_0(m)) = 0.$$

Vigon [26, Theorem 1.4] gives the following characterization of eroded processes (the result may also be found in [1, Theorem 3.11]).

Remark 3.8. A Lévy process ψ_0 with paths of unbounded variation is eroded if and only if

$$(3.8) \quad \int_0^1 x^{-1} \mathbb{P}\{\psi_0(x) \in [ax, bx]\} dx = \infty, \quad \forall a < 0 < b,$$

3.5. Randomized coterminal times. Randomized coterminal times were introduced by Millar in order to extend the set of times at which some sort of decomposition into two independent processes could take place [18]. Essentially they are last exit times from randomized sets. For example, the largest time at which the supremum of a Markov process $(\phi(x))_{x \geq 0}$ is achieved is the last exit time from the random interval $[\sup_x \phi(x), \infty)$.

In this subsection we assume $(\phi(x))_{x \geq 0}$ is a càdlàg strong Markov process with state space (E, \mathcal{E}) , a locally compact metric space (in fact we will only use state space $([0, \infty) \times \mathbb{R}, \mathcal{B}([0, \infty) \times \mathbb{R}))$). Denote by \mathcal{F}_x the sigma fields that are the right continuous completions of the natural sigma fields $\mathcal{F}_x^0 = \sigma\{\phi(y), y \leq x\}$, and let $\mathcal{F} = \bigvee_{x \geq 0} \mathcal{F}_x$. Let θ_x be the standard shift operator, so that $\phi(y)(\theta_x \omega) = \phi(x + y)(\omega)$ for every $y \geq 0$. Recall that a *random time* R is a $[0, \infty]$ -valued \mathcal{F} -measurable random variable, and that a random time T is a *terminal time* if it is optional and $T = x + T \circ \theta_x$ on $\{T > x\}$. For a random time R , define

$$\mathcal{F}(R+) := \{F \in \mathcal{F} : \text{for all } x > 0, \text{ there exists } F_x \in \mathcal{F}_x \\ \text{such that } F \cap \{R < x\} = F_x \cap \{R < x\}\}.$$

Definition 3.9. Suppose we are given

- a measure space (A, \mathfrak{U}) ,
- a family of terminal times $\{T_a\}_{a \in A}$ such that $(a, \omega) \rightarrow T_a(\omega)$ is $\mathfrak{U} \times \mathcal{F}$ -measurable,
- a measurable mapping Z from (Ω, \mathcal{F}) to (A, \mathfrak{U}) .

A random time R is a *randomized coterminal time* based on (A, \mathfrak{U}) , $\{T_a\}_{a \in A}$, Z if

- (I) for each $x \geq 0$ there is an \mathcal{F}_x -measurable A -valued random variable Z_x such that $Z = Z_x$ on the set $\{R \leq x\}$,
- (II) for each $0 \leq y < x$ there exists $B(y, x) \in \mathcal{F}_t$ such that

$$\{y \leq R < x\} = B(y, x) \cap \{T_{Z(\omega)}(\theta_x \omega) = +\infty\}.$$

Note that by (I) the Z in (II) can be replaced by Z_x .

Example 3.10. Suppose $\lim_{x \rightarrow \infty} \psi_0(x) = -\infty$, then $R = \arg \sup\{\psi_0(x) : x \geq 0\}$ is a randomized coterminal time with $(A, \mathfrak{U}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $T_a = \inf\{x > 0 : \psi_0(x) \vee \psi_0(x-) \geq a\}$, $Z = \sup_{x \geq 0} \psi_0(x)$ and $Z_x = \sup_{y \leq x} \psi_0(y)$. Property (I) is immediate since if the supremum occurs before x , then it is equal to the supremum attained by $\psi(y)$ on $[0, x]$, and to see that property (II) holds note that

$$\begin{aligned} \{y < R \leq x\} &= \{ \text{the supremum of } \psi_0 \text{ occurs in } (y, x] \} \\ &= \{ \psi_0 \text{ goes at least as high in } (y, x] \text{ as it did before time } y, \\ &\quad \{ \text{and never after } x \text{ goes as high as it did during } (y, x] \} \\ &= \{Z_{yx} \leq Z_x\} \{T_{Z_x(\omega)}(\theta_x \omega) = +\infty\}, \end{aligned}$$

where $Z_{yx} = \sup_{y < w \leq x} \psi_0(w)$, so $B(y, x) = \{Z_{yx} \leq Z_x\}$ here.

The following result is [18, Theorem 3.4] and essentially says that for a randomized coterminal time R , conditional on $Z = z$, the post R process is ‘just’ the original process conditioned on $\{T_z = +\infty\}$, and is still Markovian. Note that Z is $\mathcal{F}(R+)$ measurable by (I).

Theorem 3.11. Let $(\phi(x))_{x \geq 0}$ be a Hunt process, and R a randomized coterminal time based on (A, \mathfrak{U}) , $\{T_a\}_{a \in A}$, Z . Then for bounded Borel f ,

$$\mathbb{E}(f(\phi(R+x)) | \mathcal{F}((R+y)+)) = \int f(b) H_{x-y}(Z; \phi(R+y), db), \quad 0 < y < x,$$

where $H_x(z; a, db) := \mathbb{P}^a(\phi(x) \in db) \mathbb{P}^b(T_z = \infty) / \mathbb{P}^a(T_z = \infty)$.

The next result is a combination of [18, Proposition 5.4] and [18, (a) following Proposition 5.4], where we have trivially extended the state space to include a deterministic element as well as a Lévy process. It gives conditions under which,

conditionally given Z and $\phi(R)$, the post R process is independent of $\mathcal{F}(R+)$. The proof relies on the zero-one property of Lévy processes at local maxima or jump times.

Proposition 3.12. *Let $(\psi(x))_{x \geq 0}$ be a Lévy process and let $\phi(x) = (x, \psi(x))$ for $x \geq 0$. Let R be a randomized coterminal time for $(\phi(x))_{x \geq 0}$ based on (A, \mathfrak{U}) , $\{T_a\}_{a \in A}$, Z . Suppose that $\mathbb{P}(R \text{ is the time of a local maximum of } \psi) = \mathbb{P}(R < \infty)$ or $\mathbb{P}(R \text{ is a jump time of } \psi) = \mathbb{P}(R < \infty)$. Then conditional on Z and $\phi(R)$, the post R process is independent of $\mathcal{F}(R+)$, and it is Markov with transitions $H_x(Z; a, db)$.*

4. MAIN RESULTS

In this section we first present results in a general setting and then treat processes with paths of unbounded and bounded variation separately. Recall from Remark 3.2 that Hypothesis **B** is automatically satisfied when ψ_0 has paths of unbounded variation, and when ψ_0 has paths of bounded variation then by assumption the drift coefficient of ψ_0 is zero.

4.1. Unbounded and Bounded Variation.

Lemma 4.1. *Let ψ_0 be a two-sided Lévy process satisfying Hypotheses **A** and **B**. The Lebesgue measure of $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ is zero a.s.*

Proof. By application of Fubini's theorem and stationarity it suffices to show that $\mathbb{P}(a(x) = 0 \text{ for some } x \in \mathbb{R}) = 0$. Suppose there exists $x > 0$ such that $a(x) = 0$, then $\limsup_{h \downarrow 0} h^{-1} \psi_0(h) \leq -x < 0$. But this happens only on an event of probability zero by (3.1) or (3.3) for processes with bounded or unbounded variation respectively.

Similarly, if there exists $x < 0$ such that $a(x) = 0$, then $-\liminf_{h \uparrow 0} h^{-1} \psi_0(h) \leq -x < 0$. But this happens only on an event of probability zero by the time reversed versions of (3.1) and (3.3).

Finally, if $a(0) = 0$ then $\limsup_{h \downarrow 0} h^{-2} \psi_0(h) \leq 1 < \infty$, which by (3.5) only occurs on a set of probability zero. \square

Lemma 4.2. *Let ψ_0 be a two-sided Lévy process satisfying Hypotheses **A** and **B**. Let $a^-, a^+ \in \mathcal{A}_0$, i.e. let $u(a^-) = u(a^+) = 0$, with $a^- < a^+$ and $u(x) \neq 0$ for $a^- < x < a^+$. Then there exists $a^- < x_0 < a^+$ such that $u(x) > 0$ for all $a^- < x < x_0$ and $u(x) < 0$ for all $x_0 < x < a^+$.*

Proof. From (3.5) we have $a(0) > 0$ a.s. and hence

$$\begin{aligned} a_0^+ &:= \inf\{x \geq 0 : x \in \mathcal{A}_0\} > 0 \quad \text{a.s., and} \\ a_0^- &:= \inf\{x \geq 0 : -x \in \mathcal{A}_0\} > 0 \quad \text{a.s.,} \end{aligned}$$

where we have applied time reversal to get the second inequality.

By stationarity, it suffices to show that the claim is true for $a^+ = a_0^+$ and $a^- = a_0^-$, since any almost sure behaviour of u over the interval (a_0^-, a_0^+) must be shared by u over (a^-, a^+) for any two consecutive members $a^- < a^+$ of \mathcal{A}_0 . Define $R := \inf\{y \geq 0 : \psi_0(y-x) \vee \psi_0((y-x)-) - \psi_0(y) \vee \psi_0(y-) \leq \frac{1}{2}x^2 \text{ for all } x > 0\}$. Since R is a stopping time, (3.5) implies that $R < a(R)$ a.s., and hence $u(R) < 0$ a.s. So we cannot have $u(x) \geq 0$ for all $x \in (a_0^-, a_0^+)$. A time reversal argument then implies that we cannot have $u(x) \leq 0$ for all $x \in (a_0^-, a_0^+)$.

Since $a(x)$ is non-decreasing, u has only downwards jumps, and thus u cannot go from being negative to positive without passing through zero. Hence we have the a.s. existence of the x_0 in the claim. \square

The proof of the following Theorem is in Section 5.

Theorem 4.3. *Let ψ_0 be an abrupt two-sided Lévy process with paths of unbounded variation satisfying Hypothesis **A** or a two-sided Lévy process with paths of bounded variation satisfying Hypotheses **A** and **B** and Assumption **B**. Define*

$$T := \inf\{x \geq 0 : x \in \mathcal{A}_0\}.$$

Then $(\psi_0(T+x) - \psi_0(T))_{x \geq 0}$ is independent of $(\psi_0(T-x))_{x \geq 0}$. As a consequence, the processes $(u(T+x))_{x \geq 0}$ and $(u(T-x))_{x \geq 0}$ are independent and \mathcal{A}_0 is a regenerative set.

It is important to relate \mathcal{A}_0 to the set of Lagrangian regular points, when such points exist. As we shall see in Theorem 4.15, when ψ_0 is a two-sided Lévy process with paths of bounded variation satisfying Hypotheses **A** and **B** and Assumption **B**, \mathcal{A}_0 is exactly equal to the set of Lagrangian regular points.

4.2. Unbounded variation.

Lemma 4.4. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis **A** with paths of unbounded variation. Then ψ_0 is continuous at every point in the set $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ a.s.*

Proof. From (3.4) and a time reversal argument, it follows that almost surely for every y such that y is a jump time of ψ_0 , i.e. $\psi_0(y) \neq \psi_0(y-)$,

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) = +\infty \quad \text{and} \quad \limsup_{h \downarrow 0} h^{-1}(\psi_0(y-h) - \psi_0(y-)) = +\infty.$$

If $y = a(x)$ or $y = a(x-)$ for some x , and if y is such that say $\psi_0(y) > \psi_0(y-)$, then for every $h > 0$ we have

$$\psi_0(y) - \frac{1}{2}(x-y)^2 \geq \psi_0(y+h) - \frac{1}{2}(x-y-h)^2.$$

Therefore we would have

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(y+h) - \psi_0(y)) \leq y - x < \infty,$$

which is impossible, except on an event with probability zero. The case of a negative jump is similar, working now at the left of the jump. \square

Corollary 4.5. *Let ψ_0 be a two-sided abrupt Lévy process satisfying Hypothesis **A**. Then ψ_0 has a local supremum at every point in $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ a.s.*

Proof. Take any point $y \in \mathcal{A}$ and let x be such that $a(x) = y$. Then for every $z \geq 0$ we have

$$\psi_0(y+z) \vee \psi_0((y+z)-) - \psi_0(y) \vee \psi_0(y-) \leq \frac{1}{2}(z - (x-y))^2 - \frac{1}{2}(x-y)^2.$$

Recall from Lemma 4.4 that ψ_0 is a.s. continuous at y , thus almost surely

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-)) \leq (x-y)$$

and

$$\limsup_{h \downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x)) \leq -(x-y).$$

Theorem 3.6 then implies that ψ_0 must have a local supremum at y . \square

For abrupt Lévy processes, the shock structure is discrete.

Theorem 4.6. *Let ψ_0 be a two-sided abrupt Lévy process satisfying Hypothesis A. Then \mathcal{A} is a discrete set a.s.*

Proof. Because the random set \mathcal{A} is stationary (i.e. its law is invariant by translation), we have to prove that $\#\{[1, 2] \cap \mathcal{A}\} < \infty$ a.s. It is easy to verify that the probability that $a(x) \in [1, 2]$ for some x with $|x| > n$ goes to zero as $n \rightarrow \infty$, so it suffices in fact to establish that for each fixed n larger than some n_0 ,

$$(4.1) \quad \#\{a(x) \in [1, 2] : |x| \leq n\} < \infty \quad \text{a.s.}$$

Suppose first that $\mathbb{E}|\psi_0(1)| < \infty$. Let n_0 be large enough such that $|\mathbb{E}\psi_0(1)| < 2n_0$. Now, if a point $y \in [1, 2]$ can be expressed as $y = a(x)$ for some $x \in [-n, n]$, then

$$\psi_0(y \pm h) \vee \psi_0((y \pm h)-) < \psi_0(y) \vee \psi_0(y-) + 2nh \quad \text{for every } h \in (0, 2],$$

and since by Lemma 4.4 ψ_0 is continuous at y ,

$$\psi_0(y \pm h) < \psi_0(y) + 2nh \quad \text{for every } h \in (0, 2].$$

Defining the Lipschitz majorant of ψ_0 equivalently to how the Lipschitz minorant of ψ_0 is defined in [1], we have that if ψ_0 is continuous at y , then y is in the contact set of the $2n$ -Lipschitz majorant of ψ_0 if and only if $\psi_0(y \pm h) - \psi_0(y) < 2nh$ for all $h \in \mathbb{R}$ (note that the existence of the Lipschitz majorant follows from our assumption that $|\mathbb{E}\psi_0(1)| < 2n_0 \leq 2n$). From [1, Theorem 3.8] it follows that there are only finitely many such contact points y in the interval $[1, 2]$ almost surely. Suppose it were the case that

$$\mathbb{P}(\#\{a(x) \in [1, 2] : |x| \leq n\} = \infty) > 0.$$

Then with positive probability there would exist $y_\ell, y_r \in [1, 2]$ such that $y_\ell < y_r$ and $\#\{a(x) \in [y_\ell, y_r] : |x| \leq n\} = \infty$. Moreover, by the law of large numbers applied to the left and to the right, with positive probability there would exist such a pair with both y_ℓ and y_r in the contact set of the $2n$ -Lipschitz majorant of ψ_0 . If both y_ℓ and y_r were in the contact set of the majorant, then every element of the infinite set $\{a(x) \in [y_\ell, y_r] : |x| \leq n\}$ would also be in the contact set of the majorant, but that is an event with zero probability. Hence $\#\{a(x) \in [1, 2] : |x| \leq n\} < \infty$ a.s.

Now remove the assumption that $\mathbb{E}|\psi_0(1)| < \infty$. For each $N \in \mathbb{N}$ define the two-sided Lévy process $\tilde{\psi}_0^N$ by

$$\tilde{\psi}_0^N(x) = \begin{cases} \psi_0(x) - \sum_{\substack{0 \leq y \leq x: \\ \psi_0(y) \neq \psi_0(y-)}} (\psi_0(y) - \psi_0(y-)) 1_{|\psi_0(y) - \psi_0(y-)| > N} & \text{for } x \geq 0 \\ \psi_0(x) + \sum_{\substack{0 \leq y \leq x: \\ \psi_0(y) \neq \psi_0(y-)}} (\psi_0(y) - \psi_0(y-)) 1_{|\psi_0(y) - \psi_0(y-)| > N} & \text{for } x < 0 \end{cases}$$

so that $\tilde{\psi}_0^N$ is identical to ψ_0 but with all the jumps of magnitude greater than N removed. Let $\tilde{\mathcal{A}}^N$ be defined in the same way that \mathcal{A} is for the original process ψ_0 .

Since $\mathbb{E}|\tilde{\psi}_0^N| < \infty$ the above arguments imply that $\tilde{\mathcal{A}}^N \cap [1, 2]$ is a finite set almost surely for every N .

From the fact that $\Pi(N, \infty) < \infty$ for every $N \in \mathbb{N}$, and the hypothesis that $\psi_0(x) = o(x^2)$ a.s. as $|x| \rightarrow \infty$, it follows that almost surely there exists a random $\tilde{N} \in \mathbb{N}$ such that $\mathcal{A} \cap [1, 2] = \tilde{\mathcal{A}}^{\tilde{N}} \cap [1, 2]$. Hence $\mathcal{A} \cap [1, 2]$ is a finite set almost surely. \square

Corollary 4.7. *Let ψ_0 be a two-sided abrupt Lévy process satisfying Hypothesis A. Then $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ is closed a.s.*

The proof of the following theorem closely follows the proof of [7, Theorem 5] with Cauchy processes replaced by eroded processes.

Theorem 4.8. *Let ψ_0 be a two-sided eroded Lévy process satisfying Hypothesis A. Then with probability one there are no rarefaction intervals.*

Proof. Recall from Lemma 4.4 that jump times of ψ_0 do not belong to \mathcal{A} almost surely. Now suppose (x, x') is a rarefaction interval, that is $a(\cdot)$ stays constant on $[x, x']$; denote its value by y . As y is not a jump time of ψ_0 , we have for all $h > 0$,

$$\begin{aligned} \psi_0(y) - \frac{1}{2}(x - y)^2 &\geq \psi_0(y - h) - \frac{1}{2}(x - y + h)^2, \\ \psi_0(y) - \frac{1}{2}(x' - y)^2 &\geq \psi_0(y + h) - \frac{1}{2}(x' - y - h)^2. \end{aligned}$$

We deduce that

$$\begin{aligned} \liminf_{h \downarrow 0} h^{-1}(\psi_0(y) - \psi_0(y - h)) &\geq y - x, \\ \limsup_{h \downarrow 0} h^{-1}(\psi_0(y + h) - \psi_0(y)) &\leq y - x'. \end{aligned}$$

Since $x < x'$, we can find a rational number $q \in (y - x', y - x)$. Then y is the location of a local maximum of $(\psi_0^{(q)}(x))_{x \in \mathbb{R}}$, where $\psi_0^{(q)}(x) := \psi_0(x) - qx$, and moreover

$$(4.2) \quad \liminf_{h \downarrow 0} h^{-1}(\psi_0^{(q)}(y) - \psi_0^{(q)}(y + h)) > 0.$$

On the other hand, the family $(\psi_0^{(s)}, s \in \mathbb{Q})$ is a countable family of eroded processes. For each of these processes, with probability one, for any $s \in \mathbb{Q}$ and any location μ of a local maximum for $\psi_0^{(s)}$,

$$\liminf_{h \downarrow 0} h^{-1}(\psi_0^{(q)}(\mu) - \psi_0^{(q)}(\mu + h)) = 0.$$

We conclude that (4.2) is impossible, except on an event of probability zero, and therefore almost surely there are no rarefaction intervals. \square

4.3. Bounded variation.

Theorem 4.9. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis A with paths of bounded variation. Suppose zero is regular for $[0, \infty)$ and $(-\infty, 0]$ for $(\psi_0(x))_{x \geq 0}$, then a.s. Lagrangian regular points exist.*

Proof. We shall prove that the time of the maximum of $(\psi_0(x))_{0 \leq x \leq 1}$ has positive probability of being Lagrangian regular, and as pointed out by Bertoin in a comment before the proof of Theorem 3 of [7], it is easy to deduce from this fact that

Lagrangian regular points exist with probability one. This is because of stationarity and the asymptotic independence of the events A_0 and A_n as $n \rightarrow \infty$, where $A_n := \{\arg \sup_{n \leq x \leq n+1} \psi_0(x) \text{ is Lagrangian regular}\}$ for $n \geq 0$.

Let μ be the almost surely unique location of the maximum of $(\psi_0(x))_{0 \leq x \leq 1}$. It follows from the concave majorant theory of Pitman and Uribe-Bravo [19] that $\mu \in (0, 1)$, and that if $\bar{B} : [0, 1] \rightarrow \mathbb{R}$ denotes the concave majorant of $(\psi_0(x))_{0 \leq x \leq 1}$ then its derivative $\bar{b} = \bar{B}'$ is continuous at μ and

$$(4.3) \quad \bar{b}(\mu + h) < \bar{b}(\mu) = 0 < \bar{b}(\mu - h)$$

for every sufficiently small $h > 0$.

The rest of the argument is exactly as in the proof of Theorem 3 of [7].

(4.3) implies that the support of the Stieltjes measure $-d\bar{b}$ contains μ , and more precisely μ is neither isolated to the left nor to the right in $\text{Supp}(d\bar{b})$. Pick any $y \in \text{Supp}(d\bar{b})$ arbitrarily close to μ . Clearly, the graph of \bar{B} touches that of $(\psi_0(x))_{0 \leq x \leq 1}$ at y , so we must have $\bar{B}(y) = \psi_0(y)$ or $\bar{B}(y) = \psi_0(y-)$. In both cases, y is the location of a maximum of $x \rightarrow \psi_0(x) - \bar{b}(y)x$ on $[0, 1]$, and a fortiori y is then the unique location of the maximum of $x \rightarrow \psi_0(x) - \frac{1}{2}(y - \bar{b}(y) - x)^2$ on $[0, 1]$. Plainly, μ is also the unique location of the maximum of $x \rightarrow \psi_0(x) - \frac{1}{2}(\mu - x)^2$ on $[0, 1]$. Because $\psi_0(\mu) > \max(\psi_0(0), \psi_0(1))$, there is a positive probability that the preceding two maxima are global (i.e. on \mathbb{R}) and not only local (i.e. on $[0, 1]$). We conclude that with positive probability, $\mu \in \mathcal{A}$ and is neither isolated on its right nor on its left, and therefore is a Lagrangian regular point. \square

The next two results are due to Lachiéze-Rey [16, Theorem 4.3, Proposition 5.3] and allow us to find the behaviour of ψ_0 around points of \mathcal{A} in Proposition 4.12.

Theorem 4.10. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation. Let $\bar{C} : [0, 1] \rightarrow \mathbb{R}$ denote the concave majorant of $(\psi_0(x) - \frac{1}{2}x^2)_{x \in \mathbb{R}}$ and denote its derivative by $\bar{c} = \bar{C}'$. Then for all $a \in \mathcal{A}$, a is left isolated (resp. right isolated) in \mathcal{A} if $\bar{c}(a-) \neq -a$ (resp. $\bar{c}(a) \neq -a$).*

Proposition 4.11. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation. Suppose $y \in \mathcal{A}$ and x is such that $a(x) = y$. Then almost surely if $x < y$ then $\psi_0(y-) < \psi_0(y)$ and if $x > y$ then $\psi_0(y-) > \psi_0(y)$.*

Proposition 4.12. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis A with paths of bounded variation. Then for every $y \in \mathcal{A}$, almost surely*

- (i) if $\psi_0(y-) < \psi_0(y)$ then y is left isolated in \mathcal{A} , and
if $\psi_0(y-) > \psi_0(y)$ then y is right isolated in \mathcal{A} ;
- (ii) if y is Lagrangian regular then ψ_0 is continuous at y ;
- (iii) if $y \neq a(y)$ then y is isolated in \mathcal{A} .

Proof. (i) Suppose $\psi_0(y-) < \psi_0(y)$. Then y will be left isolated in the support of the Stieltjes measure $-d\bar{c}$, and hence will be left isolated in \mathcal{A} by Lemma 2.1. The argument is similar for the case $\psi_0(y-) > \psi_0(y)$.

(ii) Suppose ψ_0 is not continuous at y . Then either $\psi_0(y-) < \psi_0(y)$ or $\psi_0(y-) > \psi_0(y)$ and hence (i) implies that y cannot be Lagrangian regular.

(iii) By hypothesis, for any x such that $a(x) = y$, we must have $x > y$ or $x < y$. Suppose $x < y$. Proposition 4.11 implies that $\psi_0(y-) < \psi_0(y)$ and thus y will be left isolated in \mathcal{A} by (i). Moreover, $a(x) = y$ implies that $-\bar{c}(y) \leq x < y$, thus y will be right isolated in \mathcal{A} by Theorem 4.10. The argument is similar in the alternative case $x > y$. \square

Proposition 4.12 (iii) immediately leads to the following corollary.

Corollary 4.13. *Let ψ_0 be a two-sided Lévy process satisfying Hypothesis **A** with paths of bounded variation. Then the set $\{x \in \mathbb{R} : a(x) = x\}$ is closed a.s.*

Theorem 4.10 also allows us to prove the following two theorems.

Theorem 4.14. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation satisfying Hypothesis **A**, Hypothesis **B** and Assumption **B(I)**. Then the set $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ is closed a.s. and hence is equal to \mathcal{A} a.s.*

Proof. Since in the definition of $a(x)$ we take the supremum over all possible arg sups, we have that

$$(4.4) \quad \bar{c}(y-) > \bar{c}(y+h) \quad \forall h > 0 \quad \Longleftrightarrow \quad \exists x \text{ s.t. } a(x) = y.$$

Suppose y is a right accumulation point of the set $\{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$ so that there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ with $y_n \downarrow y$ and $\bar{c}(y_n-) > \bar{c}(y_n+h)$ for all $h > 0$ and hence $y \in \{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$.

Now suppose y is a left but not a right accumulation point. Then by Lemma 2.1 there exists \hat{y} such that $\bar{c}(y+h) = \bar{c}(y+)$ for all $0 \leq h < \hat{y} - y$ and such that

$$(4.5) \quad \psi_0(\hat{y}) \vee \psi_0(\hat{y}-) - \frac{1}{2}\hat{y}^2 = \bar{C}(\hat{y})$$

i.e. \hat{y} is the next contact point after y for the concave majorant of $(\psi(x) - \frac{1}{2}x^2)_{x \in \mathbb{R}}$.

Take any $q \in \mathbb{Q}$ such that $y < q < \hat{y}$. Let $\bar{C}^q : (-\infty, q] \rightarrow \mathbb{R}$ be the concave majorant of $(\psi(x) - \frac{1}{2}x^2)_{x \leq q}$ and let \bar{c}^q be its right continuous derivative, which will agree with \bar{c} on the set $(-\infty, y)$. Define

$$E^q := \{x \leq q : \bar{c}^q(x-) = -x\}.$$

Since $x \in E^q$ implies that at least one of $\limsup_{h \downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x))$ or $\limsup_{h \downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-))$ is finite, (3.5) and Fubini imply that E^q has measure zero almost surely. Also, by Theorem 4.10 we know that $y \in E^q$ a.s. since y is not isolated on the left.

The outline of the rest of the argument is as follows. For each $x \in E^q$ we will define a random time that essentially is the first time the process $(\psi_0(q+z) - \frac{1}{2}(q+z)^2)_{z \geq 0}$ is greater than or equal to the line extending out from x with slope $-x$, i.e. the same slope as the concave majorant at x . This time should be the next time the process $(\psi_0(y) - \frac{1}{2}y^2)_{y \in \mathbb{R}}$ meets its concave majorant after y , but using Assumption **B** we show that it goes strictly above that line at that time, which leads to a contradiction.

For every $x \in E^q$ define

$$T^q(x) := \inf \left\{ z \geq 0 : \psi_0(q+z) \vee \psi_0((q+z)-) - \frac{1}{2}(q+z)^2 \geq \psi_0(x) \vee \psi_0(x-) - x((q-x) + z) \right\},$$

and note that almost surely $T^q(x) > 0$ for every $x \in E^q$ since by (3.5) $\bar{C}(q) > \psi_0(q) \vee \psi_0(q-) - \frac{1}{2}q^2$ a.s. Also, by definition $T^q(y) = \hat{y}$.

By Assumption **B(I)** and the fact that E^q has measure zero a.s. it follows that a.s.

$$(4.6) \quad \psi_0(T^q(x)) - \frac{1}{2}(q + T^q(x))^2 \geq \psi_0(x) \vee \psi_0(x-) - y((q-x) + T^q(x))$$

for every $x \in E^q$ such that $T^q(x) < \infty$. But $y \in E^q$ and $T^q(y) = \hat{y}$ a.s. hence (4.5) and (4.7) would imply that $\hat{y} = \infty$, and thus y cannot be as assumed a left accumulation point and isolated on the right. Since we have shown the points not

isolated on the right are included in the set $y \in \{y \in \mathbb{R} : a(x) = y \text{ for some } x \in \mathbb{R}\}$, this concludes the proof. \square

Theorem 4.15. *Let ψ_0 be a two-sided Lévy process with paths of bounded variation satisfying Hypothesis **A**, Hypothesis **B** and Assumption **B(II)**. Then for every $y \in \mathbb{R}$, $y = a(y)$ if and only if y is a Lagrangian regular point. Hence \mathcal{A}_0 is exactly the set of Lagrangian regular points.*

Proof. Suppose $y = a(y)$ is a Lagrangian regular point, which from Theorem 4.10, is possible only if $\bar{c}(y-) = -y = \bar{c}(y)$. Since y is isolated neither on the left or the right in \mathcal{A} , Lemma 2.1 implies that

$$\bar{c}(y+h) < \bar{c}(y) = -y < \bar{c}(y-h)$$

for every $h > 0$. Thus

$$(\psi_0(y) \vee \psi_0(y-) - \frac{1}{2}y^2) - ys < (\psi_0(y+s) \vee \psi_0((y+s)-) - \frac{1}{2}(y+s)^2)$$

for all $s \neq 0$. Rearranging, we see that

$$\psi_0(y+s) \vee \psi_0((y+s)-) - \psi_0(y) \vee \psi_0(y-) - \frac{1}{2}s^2 > 0$$

for all $s \neq 0$. It follows that $y = a(y)$.

Conversely, suppose that $y = a(y)$. If y is right isolated in \mathcal{A} , then there exists $\hat{y} > y$ with $a(\hat{y}) = y$, and hence by Proposition 4.11 y is the time of a negative jump of ψ_0 . However, this would imply that $y \notin \mathcal{A}$ by the time reversed version of (3.6), and hence y is not right isolated in \mathcal{A} a.s.

If y is left isolated in \mathcal{A} we do not yet know that there necessarily exists an \hat{y} such that $\hat{y} < y$ and $a(\hat{y}) = y$, because although $y = a(y)$ implies that $\arg\sup\{\psi_0(x) - \frac{1}{2}(x-y)^2 : x \in \mathbb{R}\} = y$, the supremum may not be achieved at a unique point. Once we have shown that the supremum is unique a.s. a similar argument to the right isolated case above would show that y is not left isolated in y a.s. and hence that $y = a(y)$ implies that y is a Lagrangian regular point.

Suppose that $\arg\sup\{\psi_0(x) - \frac{1}{2}(x-y)^2 : x \in \mathbb{R}\} = y$ and there exists $\hat{y} < y$ such that $\psi_0(\hat{y}) \vee \psi_0(\hat{y}-) - \frac{1}{2}(\hat{y}-y)^2 = \psi_0(y) \vee \psi_0(y-)$, i.e. suppose that the supremum is not unique, and suppose further that \hat{y} is maximal among points for where the supremum is attained other than y .

Take any $q \in \mathbb{Q}$ with $\hat{y} < q < y$. The remainder of the argument is a time reversed analogue of the argument used in the proof of Theorem 4.14 with a slightly expanded definition of E^q . Let $\bar{C}_q : [q, \infty) \rightarrow \mathbb{R}$ be the concave majorant of $(\psi(x) - \frac{1}{2}x^2)_{x \geq q}$ and let \bar{c}_q be its right continuous derivative, which will agree with \bar{c} on the set (y, ∞) . Define

$$E_q := \{x \geq q : \bar{c}_q(x-) \leq -x, \bar{c}_q(x) \geq x\}.$$

Since $x \in E_q$ implies that at least one of $\limsup_{h \downarrow 0} h^{-1}(\psi_0(x+h) - \psi_0(x))$ or $\limsup_{h \downarrow 0} h^{-1}(\psi_0(x-h) - \psi_0(x-))$ is finite, (3.5) and Fubini imply that E_q has measure zero almost surely. Also, $a(y) = y$ it follows that $y \in E_q$.

For every $x \in E_q$ define

$$T_q(x) := \inf \left\{ z \geq 0 : \psi_0(q-z) \vee \psi_0((q-z)-) - \frac{1}{2}(q-z)^2 \geq \psi_0(x) \vee \psi_0(x-) - x((q-x)-z) \right\},$$

and note that almost surely $T_q(x) > 0$ for every $x \in E_q$ since by (3.5) $\bar{C}(q) > \psi_0(q) \vee \psi_0(q-) - \frac{1}{2}q^2$ a.s. Also, by definition $T_q(y) = \hat{y}$.

By Assumption **B**(II) (its time reversed version – see Remark 3.3(ii)) and the fact that E_q has measure zero a.s. it follows that a.s.

$$(4.7) \quad \psi_0(T_q(x)) - \frac{1}{2}(q - T_q(x))^2 \geq \psi_0(x) \vee \psi_0(x-) - y((q - x) - T_q(x))$$

for every $x \in E_q$ such that $T_q(x) < \infty$. But $y \in E_q$ and $T_q(y) = \hat{y}$ a.s. hence (4.5) and (4.7) would imply that $\hat{y} = -\infty$, and thus \hat{y} cannot exist as assumed. \square

5. PROOF OF THEOREM 4.3

5.1. Facts relating to the first non-negative element of \mathcal{A}_0 . In this section, we prove some results relating to the first non-negative element of \mathcal{A}_0 when ψ_0 is a non-random càdlàg function satisfying $\lim_{|x| \rightarrow \infty} x^{-2}\psi_0(x) = 0$. Define

$$\begin{aligned} \mathbf{t} &:= \inf\{y \geq 0 : a(y) = y\} \\ &= \inf\left\{y \geq 0 : \arg \sup\left\{\psi_0(x) - \frac{1}{2}(x - y)^2 : x \in \mathbb{R}\right\} = y\right\}, \\ &= \inf\left\{y \geq 0 : \psi_0(y - x) \vee \psi_0((y - x)-) - \psi_0(y) \vee \psi_0(y-) \leq \frac{1}{2}x^2 \text{ for all } x > 0 \text{ and} \right. \\ &\quad \left. \psi_0(y + x) \vee \psi_0((y + x)-) - \psi_0(y) \vee \psi_0(y-) < \frac{1}{2}x^2 \text{ for all } x > 0\right\} \end{aligned}$$

The last equality is because of the convention that if the arg sup above is not unique we take it to be the supremum over all suitable arguments. Define further

$$\begin{aligned} \mathbf{r} &:= \inf\left\{y \geq 0 : \psi_0(y - x) \vee \psi_0((y - x)-) - \psi_0(y) \vee \psi_0(y-) \leq \frac{1}{2}x^2 \text{ for all } x > 0\right\}, \\ \mathbf{s} &:= \inf\left\{y \geq \mathbf{r} : \psi_0(y + x) \vee \psi_0((y + x)-) - \psi_0(y) \vee \psi_0(y-) < \frac{1}{2}x^2 \text{ for all } x > 0\right\}. \end{aligned}$$

Note that $0 \leq \mathbf{r} \leq \mathbf{s} \leq \mathbf{t}$.

Lemma 5.1. *Let ψ_0 be any càdlàg function with $\lim_{|x| \rightarrow \infty} x^{-2}\psi_0(x) = 0$. Then the infimum in the definition of \mathbf{s} is achieved, that is,*

$$(5.1) \quad \psi_0(\mathbf{s} + x) \vee \psi_0((\mathbf{s} + x)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) < \frac{1}{2}x^2 \text{ for all } x > 0.$$

Proof. Suppose that (5.1) did not hold. Then by the definition of \mathbf{s} there would exist a strictly decreasing sequence $\{\mathbf{s}_n\}_{n \geq 0}$ such that $\lim_n \mathbf{s}_n = \mathbf{s}$ and

$$(5.2) \quad \psi_0(\mathbf{s}_n + x) \vee \psi_0((\mathbf{s}_n + x)-) - \psi_0(\mathbf{s}_n) \vee \psi_0(\mathbf{s}_n-) < \frac{1}{2}x^2 \text{ for all } x > 0$$

for every $n \geq 0$.

For $n \geq 1$, (5.2) with $x = \mathbf{s}_{n-1} - \mathbf{s}_n$ gives

$$\psi_0(\mathbf{s}_n) \vee \psi_0(\mathbf{s}_n-) > \psi_0(\mathbf{s}_{n-1}) \vee \psi_0(\mathbf{s}_{n-1}-) - \frac{1}{2}(\mathbf{s}_{n-1} - \mathbf{s}_n)^2$$

and thus since $\sum_{m=1}^n (\mathbf{s}_{m-1} - \mathbf{s}_m)^2 \leq (\mathbf{s}_0 - \mathbf{s}_n)^2$ we have

$$\psi_0(\mathbf{s}_n) \vee \psi_0(\mathbf{s}_n-) > \psi_0(\mathbf{s}_0) \vee \psi_0(\mathbf{s}_0-) - \frac{1}{2}(\mathbf{s}_0 - \mathbf{s}_n)^2.$$

By right continuity of $\psi_0(\cdot)$ at \mathbf{s} , recalling that $\lim_n \mathbf{s}_n = \mathbf{s}$ we may take the limit as $n \rightarrow \infty$ to get that

$$(5.3) \quad \psi_0(\mathbf{s}) \geq \psi_0(\mathbf{s}_0) \vee \psi_0(\mathbf{s}_0-) - \frac{1}{2}(\mathbf{s}_0 - \mathbf{s})^2.$$

Now, since we have assumed that (5.1) does not hold, there exists $x^* > 0$ such that

$$\psi_0(\mathbf{s} + x^*) \vee \psi_0((\mathbf{s} + x^*)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \geq \frac{1}{2}(x^*)^2,$$

and moreover without loss of generality we can assume that \mathbf{s}_0 is such that $\mathbf{s}_0 < \mathbf{s} + x^*$. But then starting from (5.3) we get

$$\begin{aligned} \psi_0(\mathbf{s}_0) \vee \psi_0(\mathbf{s}_0-) &\leq \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) + \frac{1}{2}(\mathbf{s}_0 - \mathbf{s})^2 \\ &\leq \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) + \frac{1}{2}(x^*)^2 - \frac{1}{2}((\mathbf{s} + x^*) - \mathbf{s}_0)^2 \\ &\leq \psi_0(\mathbf{s} + x^*) \vee \psi_0((\mathbf{s} + x^*)-) - \frac{1}{2}((\mathbf{s} + x^*) - \mathbf{s}_0)^2, \end{aligned}$$

which contradicts (5.2) with $n = 0$ and $x = \mathbf{s} + x^* - \mathbf{s}_0$. \square

Lemma 5.2. *Let ψ_0 be any càdlàg function with $\lim_{|x| \rightarrow \infty} x^{-2}\psi_0(x) = 0$. Then $\mathbf{s} = \mathbf{t}$.*

Proof. Recall that $0 \leq \mathbf{r} \leq \mathbf{s} \leq \mathbf{t}$. We will show that at \mathbf{s} the conditions of \mathbf{r} are still satisfied, i.e.

$$(5.4) \quad \psi_0(\mathbf{s} - x) \vee \psi_0((\mathbf{s} - x)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \leq \frac{1}{2}x^2 \text{ for all } x > 0$$

which combined with (5.1) implies that $\mathbf{s} \geq \mathbf{t}$ and hence $\mathbf{s} = \mathbf{t}$.

Suppose first that $\mathbf{r} = \mathbf{s}$, then clearly (5.4) is satisfied and hence $\mathbf{s} = \mathbf{t}$. Assume therefore that $\mathbf{r} < \mathbf{s}$. We will begin by showing that (5.4) holds for all $0 < x \leq \mathbf{s} - \mathbf{r}$. It suffices to show that if we define

$$(5.5) \quad \tau := \arg \sup \{ \psi_0(\mathbf{s} - y) \vee \psi_0((\mathbf{s} - y)-) - \frac{1}{2}y^2 : 0 \leq y \leq \mathbf{s} - \mathbf{r} \}$$

then we must have $\tau = 0$. Well, (5.5) implies that

$$\psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) - \frac{1}{2}\tau^2 \geq \psi_0(\mathbf{s} - y) \vee \psi_0((\mathbf{s} - y)-) - \frac{1}{2}y^2$$

for all $0 \leq y \leq \tau$. Making the change of variables $y = \tau - x$, we see that

$$\psi_0(\mathbf{s} - \tau + x) \vee \psi_0((\mathbf{s} - \tau + x)-) - \psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) \leq \frac{1}{2}x^2 - x\tau$$

for all $0 \leq x \leq \tau$. Suppose that $\tau > 0$, so that

$$\psi_0(\mathbf{s} - \tau + x) \vee \psi_0((\mathbf{s} - \tau + x)-) - \psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) < \frac{1}{2}x^2$$

for all $0 < x \leq \tau$. Combined with (5.1) this would imply that

$$\psi_0(\mathbf{s} - \tau) \vee \psi_0((\mathbf{s} - \tau)-) + \frac{1}{2}x^2 > \psi_0((\mathbf{s} - \tau) + x) \text{ for all } x > 0.$$

But then since $\mathbf{s} - \tau \geq \mathbf{r}$, the definition of \mathbf{s} would then imply that $\mathbf{s} \leq \mathbf{s} - \tau < \mathbf{s}$, a clear contradiction. Hence $\tau = 0$ as required.

It remains to show that (5.4) holds for all $x > \mathbf{s} - \mathbf{r}$. Applying (5.4) at $x = \mathbf{s} - \mathbf{r}$ we see that

$$\psi_0(\mathbf{r}) \vee \psi_0(\mathbf{r}-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \leq \frac{1}{2}(\mathbf{s} - \mathbf{r})^2.$$

From the definition of \mathbf{r} ,

$$\psi_0(\mathbf{r} - y) \vee \psi_0((\mathbf{r} - y)-) - \psi_0(\mathbf{r}) \vee \psi_0(\mathbf{r}-) \leq \frac{1}{2}y^2$$

for all $y > 0$, and hence

$$\psi_0(\mathbf{r} - y) \vee \psi_0((\mathbf{r} - y)-) - \psi_0(\mathbf{s}) \vee \psi_0(\mathbf{s}-) \leq \frac{1}{2}y^2 + \frac{1}{2}(\mathbf{s} - \mathbf{r})^2 < \frac{1}{2}((\mathbf{s} - \mathbf{r}) + y)^2$$

for all $y > 0$. Applying the change of variables $x = (\mathbf{s} - \mathbf{r}) + y$ shows that (5.4) holds for all $x > \mathbf{s} - \mathbf{r}$ and hence completes the proof. \square

Define $\mathbf{r}_0 := \mathbf{r}$, and for $k \geq 0$, define

$$\mathbf{r}_{k+1} := \mathbf{r}_k + \arg \sup \{ \psi_0(\mathbf{r}_k + x) \vee \psi_0((\mathbf{r}_k + x)-) - \frac{1}{2}x^2 : x \geq 0 \},$$

where if the $\arg \sup$ above is not unique we take it to be the supremum over all suitable arguments.

Lemma 5.3. *Let ψ_0 be any càdlàg function with $\lim_{|x| \rightarrow \infty} x^{-2}\psi_0(x) = 0$. Then $\mathbf{r}_k \rightarrow \mathbf{t}$.*

Proof. Note first that $\mathbf{r}^* := \lim_k \mathbf{r}_k$ exists since \mathbf{r}_k is an increasing sequence. If there were a $k \geq 0$ such that $\mathbf{r}_k = \mathbf{t}$, then necessarily $\mathbf{r}_j = \mathbf{t}$ for all $j \geq k$, thus we henceforth assume there is no such k .

Suppose that there exists a $k \geq 0$ such that $\mathbf{r}_k < \mathbf{t} < \mathbf{r}_{k+1}$, then

$$(5.6) \quad \psi_0(\mathbf{t}) \vee \psi_0(\mathbf{t}-) - \frac{1}{2}(\mathbf{t} - \mathbf{r}_k)^2 \leq \psi_0(\mathbf{r}_{k+1}) \vee \psi_0(\mathbf{r}_{k+1}-) - \frac{1}{2}(\mathbf{r}_{k+1} - \mathbf{r}_k)^2.$$

From the definition of \mathbf{t} it follows that

$$\psi_0(\mathbf{r}_{k+1}) \vee \psi_0(\mathbf{r}_{k+1}-) - \psi_0(\mathbf{t}) \vee \psi_0(\mathbf{t}-) - \frac{1}{2}(\mathbf{r}_{k+1} - \mathbf{t})^2 < 0.$$

Thus if equality held in (5.6) it would be the case that

$$(\mathbf{r}_{k+1} - \mathbf{r}_k)^2 < (\mathbf{r}_{k+1} - \mathbf{t})^2 + (\mathbf{t} - \mathbf{r}_k)^2 = (\mathbf{r}_{k+1} - \mathbf{r}_k)^2 - 2(\mathbf{r}_{k+1} - \mathbf{t})(\mathbf{t} - \mathbf{r}_k),$$

and hence the inequality in (5.6) must be strict. (5.6) then implies that

$$\begin{aligned} \psi_0(\mathbf{r}_{k+1}) \vee \psi_0(\mathbf{r}_{k+1}-) - \psi_0(\mathbf{t}) \vee \psi_0(\mathbf{t}-) - \frac{1}{2}(\mathbf{r}_{k+1} - \mathbf{t})^2 \\ > (\mathbf{r}_{k+1} - \mathbf{r}_k)^2 - (\mathbf{r}_{k+1} - \mathbf{t})^2 - (\mathbf{t} - \mathbf{r}_k)^2 > 0, \end{aligned}$$

which contradicts the definition of \mathbf{t} , and hence there is no k such that $\mathbf{r}_k < \mathbf{t} < \mathbf{r}_{k+1}$. Thus $\mathbf{r}^* \leq \mathbf{t}$.

Suppose $\mathbf{r}^* < \mathbf{t}$, then $\mathbf{r}^* < \mathbf{s}$ by Lemma 5.2, and hence there exists $r_+ > 0$ such that

$$\psi_0(\mathbf{r}^* + r_+) - \psi_0(\mathbf{r}^*) \vee \psi_0(\mathbf{r}^*-) - \frac{1}{2}r_+^2 > 0.$$

Let $r_- > 0$ be such that

$$\frac{1}{2}(r_+ + r_-)^2 = \psi_0(\mathbf{r}^* + r_+) - \psi_0(\mathbf{r}^*) \vee \psi_0(\mathbf{r}^*-).$$

Then for all k large enough such that $\mathbf{r}_k > \mathbf{r}^* - r_-$ we have

$$\frac{1}{2}((\mathbf{r}^* + r_+) - \mathbf{r}_k)^2 < \psi_0(\mathbf{r}^* + r_+) - \psi_0(\mathbf{r}^*) \vee \psi_0(\mathbf{r}^*-)$$

and hence

$$\mathbf{r}_{k+1} > \mathbf{r}_k + ((\mathbf{r}^* + r_+) - \mathbf{r}_k) = \mathbf{r}^* + r_+ > \mathbf{r}^*,$$

which is clearly a contradiction. Thus we can conclude that $\mathbf{r}^* = \mathbf{t}$. \square

5.2. Randomized coterminal times relating first non-negative element of \mathcal{A}_0 . In this section we will use the notation of Definition 3.9 when checking if a given random time is a randomized coterminal time.

Lemma 5.4. *Let ψ_0 be a real valued strong Markov process. Define a sequence of random times by $R_0 = 0$, and for $k \geq 0$,*

$$\begin{aligned} R_{k+1} &:= R_k + \arg \sup \left\{ \psi_0(R_k + x) \vee \psi_0((R_k + x)-) - \psi_0(R_k) \vee \psi_0(R_k-) - \frac{1}{2}x^2 : x \geq 0 \right\} \\ &= R_k + \arg \sup \left\{ \psi_0(R_k + x) \vee \psi_0((R_k + x)-) - \frac{1}{2}x^2 : x \geq 0 \right\}, \end{aligned}$$

where if the $\arg \sup$ above is not unique we take it to be the supremum over all suitable arguments. Define ϕ to be the process $(\phi(x))_{x \geq 0}$, with

$$\phi(x) = (\phi_1(x), \phi_2(x)) := (x, \psi_0(x))$$

for all $x \geq 0$. Then R_k is a randomized coterminal time for ϕ for each $k \geq 1$.

Proof. Let $A = \mathbb{R}^3$, $\mathfrak{U} = \mathcal{B}(\mathbb{R}^3)$ and let $Z = (R_{k-1}, R_k, \psi_0(R_k) \vee \psi_0(R_k-))$. Let $R_0^{(x)} = 0$ and for $k \geq 0$, if $R_k^{(x)} < x$ then let

$$R_{k+1}^{(x)} := R_k^{(x)} + \arg \sup \left\{ \psi_0(R_k^{(x)} + y) \vee \psi_0((R_k^{(x)} + y)-) - \frac{1}{2}y^2 : 0 \leq y \leq x - R_k^{(x)} \right\},$$

but if $R_k^{(x)} = x$ then let $R_{k+1}^{(x)} = x$. Let $Z_x = (R_{k-1}^{(x)}, R_k^{(x)}, \psi_0(R_k^{(x)}) \vee \psi_0(R_k^{(x)}-))$, so that Z_x is an \mathcal{F}_x -measurable A -valued random variable as required. Finally, recalling that $(\phi_1(x), \phi_2(x)) = (x, \psi_0(x))$, define the family of terminal times $\{T_a\}_{a \in A}$ by

$$T_{(a_1, a_2, a_3)} := \inf \{x > 0 : \phi_2(x) \vee \phi_2(x-) - a_3 + \frac{1}{2}(a_1 - a_2)^2 \geq \frac{1}{2}(\phi_1(x) - a_1)^2\}.$$

(I) and (II) follow once we define $B(y, x) := \{y \leq R_k^{(x)} < x\}$. \square

Lemma 5.5. *Let ψ_0 be a real valued strong Markov process and define*

$$F := \inf \{x \geq 0 : \psi_0(x+s) \vee \psi_0((x+s)-) - \psi_0(x) \vee \psi_0(x-) < \frac{1}{2}s^2 \text{ for all } s > 0\}.$$

Define ϕ to be the process $(\phi(x))_{x \geq 0}$, with

$$\phi(x) = (\phi_1(x), \phi_2(x)) := (x, \psi_0(x))$$

for all $x \geq 0$. Then F is a randomized coterminal time for ϕ .

Proof. Let $A = \mathbb{R}^2$, $\mathfrak{U} = \mathcal{B}(\mathbb{R}^2)$, $Z = (F, \psi_0(F) \vee \psi_0(F-))$ and $Z_x = (F_x, \psi_0(F_x) \vee \psi_0(F_x-))$, where

$$F_x := \inf \{0 \leq y \leq x : \psi_0(y+s) \vee \psi_0((y+s)-) - \psi_0(y) \vee \psi_0(y-) \leq \frac{1}{2}s^2 \text{ for all } 0 < s \leq x-y\}.$$

It follows that Z_x is an \mathcal{F}_x -measurable A -valued random variable. Finally, recalling that $(\phi_1(x), \phi_2(x)) = (x, \psi_0(x))$, define the family of terminal times $\{T_a\}_{a \in A}$ by

$$T_{(a_1, a_2)} := \inf \{x > 0 : \phi_2(x) \vee \phi_2(x-) - a_2 \geq \frac{1}{2}(\phi_1(x) - a_1)^2\}.$$

By definition,

$$\psi_0(F+s) \vee \psi_0((F+s)-) - \psi_0(F) \vee \psi_0(F-) < \frac{1}{2}s^2$$

for all $s > 0$, and

$$\psi_0(F_x+s) \vee \psi_0((F_x+s)-) - \psi_0(F_x) \vee \psi_0(F_x-) < \frac{1}{2}s^2$$

for all $0 < s \leq x - F_x$. In particular, if $F \leq x$, then

$$\psi_0(F) \vee \psi_0(F-) - \psi_0(F_x) \vee \psi_0(F_x-) < \frac{1}{2}(F - F_x)^2.$$

Hence we see that on the set $\{F \leq x\}$,

$$\psi_0(F_x+s) \vee \psi_0((F_x+s)-) - \psi_0(F_x) \vee \psi_0(F_x-) < \frac{1}{2}s^2$$

for all $s > 0$, which implies that $F \leq F_x$. However, $F \geq F_x$ by definition, and therefore $F_x = F$ on the set $\{F \leq x\}$. Thus (I) is satisfied.

If we define $B(y, x) := \{y \leq F_x < x\}$, then clearly

$$\{y \leq F < x\} = B(y, x) \cap \{T_{Z_x(\omega)}(\theta_x \omega) = +\infty\} = B(y, x) \cap \{T_{Z(\omega)}(\theta_x \omega) = +\infty\},$$

and hence (II) is satisfied. \square

Corollary 5.6. *Let ψ_0 be a Lévy process and define F as in Lemma 5.5. Suppose that ψ_0 is continuous at F . Then for any (x_1, \dots, x_n) with $x_i > 0$ for $i = 1, \dots, n$, the joint law of $(\psi_0(F+x_i) - \psi_0(F))_{i=1, \dots, n}$ depends only on (x_1, \dots, x_n) .*

Proof. From Theorem 3.11 we know that the joint law of $(\psi_0(F + x_i))_{i=\{1,\dots,n\}}$ depends only on (x_1, \dots, x_n) and $Z = (F, \psi_0(F))$. Moreover we can think of the post F process $(\psi_0(F + x))_{x \geq 0}$ as the original process started at $\psi_0(F)$ but conditioned to remain below a half parabola with its minimum at $\psi_0(F) \vee \psi_0(F-) = \psi_0(F)$. Then by the spatial homogeneity of Lévy processes, the joint law of $(\psi_0(F + x_i) - \psi_0(F))_{i=\{1,\dots,n\}}$ cannot depend on $\psi_0(F)$, and by the temporal homogeneity of Lévy processes it cannot depend on F either. Thus the joint law of $(\psi_0(F + x_i) - \psi_0(F))_{i=\{1,\dots,n\}}$ can depend only on (x_1, \dots, x_n) . \square

5.3. Proof of Theorem 4.3.

Proof. (Theorem 4.3) Recall from the statement of the theorem that $T := \inf\{x \geq 0 : x \in \mathcal{A}_0\}$ and hence $T = \inf\{x \geq 0 : a(x) = x\}$. From Corollary 4.13 in the bounded variation case or Theorem 4.6 in the abrupt case, we know that the set $\{x \in \mathbb{R} : a(x) = x\}$ is closed a.s. and hence $a(T) = T$ a.s.

If ψ_0 has paths of unbounded variation, then Lemma 4.4 then implies that ψ_0 is continuous at T a.s. If ψ_0 has paths of bounded variation Theorem 4.15 implies that T is a Lagrangian regular point a.s. and then Proposition 4.12(ii) implies that ψ_0 is continuous at T a.s.

Define two further random variables R and S by

$$R := \inf \left\{ y \geq 0 : \psi_0(y - x) \vee \psi_0((y - x)-) - \psi_0(y) \vee \psi_0(y-) \leq \frac{1}{2}x^2 \text{ for all } x > 0 \right\},$$

$$S := \inf \left\{ y \geq R : \psi_0(y + x) \vee \psi_0((y + x)-) - \psi_0(y) \vee \psi_0(y-) < \frac{1}{2}x^2 \text{ for all } x > 0 \right\}.$$

Note that $0 \leq R \leq S \leq T$. Note also that by the strong Markov property applied at the stopping time R , it follows that $S - R$ has the same law as F in Lemma 5.5.

Lemma 5.2 tells us that $S = T$ a.s., and thus $T = R + (S - R)$ a.s. Since R is a stopping time, $(\psi_0(R + x) - \psi_0(R))_{x \geq 0}$ is independent of $(\psi_0(R - x))_{x \geq 0}$ and has the same law as $(\psi_0(x))_{x \geq 0}$. Since $S - R$ has the same law as F in Lemma 5.5, we only need to show that the process $(\psi_0(F + x) - \psi_0(F))_{x \geq 0}$ is independent of $(\psi_0(x))_{0 \leq x \leq F}$ when ψ_0 is a.s. continuous at F , and when we can further assume that

$$(5.7) \quad \psi_0(x) - \frac{1}{2}x^2 \leq 0 \text{ for all } x \leq 0.$$

By continuity of ψ_0 at F we only need to show that $(\psi_0(F + x) - \psi_0(F))_{x \geq 0}$ is independent of $(\psi_0(x))_{0 \leq x < F}$. Moreover, Corollary 5.6 implies that the law of $(\psi_0(F + x) - \psi_0(F))_{x \geq 0}$ cannot depend on F or $\psi_0(F)$, hence it is enough to show that

$$(5.8) \quad (\psi_0(F + x) - \psi_0(F))_{x \geq 0} \text{ is independent of } (\psi_0(x))_{0 \leq x < F},$$

conditionally given F and $\psi_0(F)$.

Suppose first that ψ_0 has paths of unbounded variation and is abrupt. From Corollary 4.5 ψ_0 must have a local maximum at F , and from Lemma 5.5 we know that F is a randomized coterminal time for the process $(x, \psi_0(x))_{x \geq 0}$, hence by Proposition 3.12 it follows that $(F + x, \psi_0(F + x) - \psi_0(F))_{x \geq 0}$ is independent of $(x, \psi_0(x))_{0 \leq x \leq F}$ conditionally given $(F, \psi_0(F))$. Hence we have (5.8).

Now suppose that ψ_0 has paths of bounded variation. Define a sequence of random times by $R_0 = 0$, and

$$R_{k+1} := R_k + \arg \sup \left\{ \psi_0(R_k + x) \vee \psi_0((R_k + x)-) - \frac{1}{2}x^2 : x \geq 0 \right\}$$

for $k \geq 0$.

From Lemma 5.3 and the fact that $S - R$ has the same law as F , we have that $R_k \rightarrow F$ a.s. Suppose we have shown that for each $k \geq 1$, the process $(\psi_0(R_k + x))_{x \geq 0}$ is independent of $(\psi_0(x))_{0 \leq x \leq R_k}$ conditionally given R_k and $\psi_0(R_k)$. If $R_k = R_{k+1}$ for some k , then $R_k = T$ and we are done. Thus assume that $R_k < R_{k+1}$ for every k . We have that $R_k \rightarrow F$ a.s., and the a.s. continuity of ψ_0 at F implies that $\psi_0(R_k) \rightarrow \psi_0(F)$. Thus the process $(\psi_0(F + x))_{x \geq 0}$ is independent of $(\psi_0(x))_{0 \leq x < R_k}$ conditionally given F and $\psi_0(F)$, and (5.8) follows.

It remains to show that for each $k \geq 1$, the process $(\psi_0(R_k + x))_{x \geq 0}$ is independent of $(\psi_0(x))_{0 \leq x \leq R_k}$ conditionally given R_k and $\psi_0(R_k)$. Note that under (5.7), $a(R_k) = R_{k+1}$ for every $k \geq 0$. Since we have assumed that $R_k < R_{k+1}$ for every $k \geq 0$, it follows from Proposition 4.11 that R_{k+1} is a positive jump time of ψ_0 for every $k \geq 0$. From Lemma 5.4 we know that R_k is a randomized coterminal time for the process $(x, \psi_0(x))_{x \geq 0}$, hence by Proposition 3.12 it follows that $(R_k + x, \psi_0(R_k + x) - \psi_0(R_k))_{x \geq 0}$ is independent of $(x, \psi_0(x))_{0 \leq x \leq R_k}$ conditionally given $(R_k, \psi_0(R_k))$. \square

Remark 5.7. For processes with bounded variation satisfying Hypotheses **A** and **B**, Giraud's proof of the regenerativity of the set of Lagrangian regular points [13, Theorem 2] when ψ_0 is a stable Lévy process with stability index $\alpha \in (1/2, 1)$ could also be used to prove Theorem 4.3. Hypothesis **B** ensures that equation (7) of [13] holds appropriately, and Theorem 4.15 ensures that the first sentence of Lemma 4 of [13] is true. Those are the only two results needed in that proof.

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